Expected Value Theory I

Ryan Doody

March 1, 2022

Rational Choices

How should you evaluate your options when you are uncertain of their consequences? Two considerations seem to matter: (i) How good or bad are the various potential outcomes? (ii) How likely are those potential outcomes to obtain?

The *expected value* of a wager is (roughly) the amount you'd expect to win, on average, in the long run.

Consider an option's **expected value:**

> Let $L = \{\langle p_1, \$x_1\rangle, \langle p_2, \$x_2\rangle, \dots\}$ be a wager that pays $\$x_1$ with probability p_1, $\$x_2$ with probability p_2, etc. The *expected value* of wager L is the weighted average of its potential payoffs, where the weights correspond the probability of the wager paying out that amount.

The average of $a_1, \dots, a_n$ is

$$\frac{a_i + \dots a_n}{n} = \sum_{i=1}^{n} (\frac{1}{n}) \cdot a_i$$

Here, the weights—$1/n$—are all the same. We can get a *weighted* average by changing the weights (just so long as they sum to 1).

$$\begin{aligned} EV(L) &= \sum_i p_i \cdot x_i \\ &= p_1 \cdot x_1 + p_2 \cdot x_2 + \dots \end{aligned}$$

Claim: You rationally ought to value wagers in accordance with their expected values (i.e., prefer wagers with higher expected values; be indifferent when they have the same).

Is this right? If so, what supports valuing wagers like this rather than some other way?

A Brief Interlude: The Probability Axioms

An option's expected value depends on probabilities. What is probability?

Any function satisfying these axioms is a probability function. But the axioms don't settle how to *interpret* what probabilities are.

On some views, probabilities are long-run frequencies (*Frequentism*).

On other views, probabilities are subjective degrees of belief (*Bayesianism*).

Existential *risks* are probabilities. But in which sense?

Normality
Every proposition is assigned a number between 0 and 1.

$$0 \leq \Pr(X) \leq 1 \tag{1}$$

Certainty
Any proposition $\top$ that is certain to be true (e.g., a logical truth) is assigned probability 1.

$$\Pr(\top) = 1 \tag{2}$$

Additivity
If propositions X and Y are *mutually exclusive*, then the probability of their disjunction is equal to the sum of their probabilities.

Two propositions are *mutually exclusive* just in case they cannot *both* be true.

$$\text{If } X \& Y \text{ are mutually exclusive, } \Pr(X \vee Y) = \Pr(X) + \Pr(Y) \tag{3}$$

Conditional Probability: Let $\Pr(X \mid Y)$ be the probability of X *conditional* on Y. (Roughly: the probability that X is the case on the assumption that Y is the case.)

$$\Pr(X \mid Y) = \frac{\Pr(X \wedge Y)}{\Pr(Y)} \tag{4}$$

Should You Maximize Expected (Monetary) Value?

The St. Petersburg Paradox. I will flip a fair coin until it comes up heads; if the first time it comes up heads is the n^{th} toss then I will pay you $\$2^n$. What's the most you'd be willing to pay for this wager? What is its expected monetary payout?

This problem was first raised by Nicholas Bernoulli. It inspired Gabriel Cramer and Daniel Bernoulli (Nicholas' brother) to solve the paradox by arguing that money has diminishing marginal value.

Toss	**Payout** (x_i)	**Probability** (p_i)
H	\$2	$1/2$
TH	\$4	$1/4$
TTH	\$8	$1/8$
$\vdots$	$\vdots$	$\vdots$
$\underbrace{T\ldots TH}_{n}$	$\$2^n$	$1/2^n$
$\vdots$	$\vdots$	$\vdots$

What's its expected monetary payout?

$$\sum_i p_i \cdot x_i = 1/2 \cdot \$2 + 1/4 \cdot \$4 + 1/8 \cdot \$8 + \cdots + 1/2^n \cdot \$2^n + \ldots$$

$$= \$1 + \$1 + \$1 + \cdots + \$1 + \cdots = \infty$$

Money has Diminishing Marginal Utility: If $x > y$, the difference in value between having $\$x$ and having $\$\,(x+y/2)$ is greater than the difference in value between having $\$\,(x+y/2)$ and having $\$y$.

Money has declining marginal utility, for example, if $2u(\$x) > u(\$2x)$.

If $2u(\$x) > u(\$2x)$, then $2u(\$2^n) > u(\$2^{n+1})$.

And, because $\sum_n a_n$ converges if, for all n, $\frac{a_{n+1}}{a_n} < 1$, the expected utility of the St. Petersburg wager ($= \sum_n \frac{1}{2^n} \cdot u(\$2^n)$) converges to a finite amount.

Cramer/Bernoulli Response: Money has diminishing marginal "utility" and it is expected *utility*—not monetary payouts—that rationality requires us to maximize.

Subjective Expected Utility Theory

Let $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_k}$ be mutually exclusive and exhaustive *acts*.

Let $S_1, S_2, \ldots, S_n$ be a mutually exclusive and exhaustive *states*.

Every pair of acts and states $\langle \mathbf{A_i}, S_j \rangle$ corresponds to an *outcome*: $O[\mathbf{A_i}, S_j] = (A_i \wedge S_j)$.

Let $\Pr(S_j)$ be the *probability* that the world is in state S_j, and

Let $u(A_i \wedge S_j)$ be the *subjective degree of value* assigned to the outcome resulting from act $\mathbf{A_i}$ in state S_j.

Daniel Bernoulli proposed that utility is a logarithmic function of money (e.g., $u(x) = \log(x)$), but why think utility is *objective*? Can't different people value money in different ways? And don't we value things other than money?

Proposal: Rationality requires you to maximize the expectation of your *subjective* utility function.

Worry: What is your subjective utility function like? Can you introspect what precise utility you assign to various outcomes? If not, how could it be measured?

Frank Ramsey's Answer: Quantify how much someone values something by considering how much they'd risk to get it.

The **expected utility** of **A** is:

$$EU(\mathbf{A}) = \sum_i \Pr(S_i) \cdot u(\mathbf{A} \wedge S_i)$$

Issues: Can't our actions change what probabilities we should assign? What about (genuine) risk-aversion? Is that irrational? What probabilities should we assign when we are deeply uncertain about what might happen?

DECISION MATRIX

	S_1	S_2	$\ldots$	S_n
$\mathbf{A_1}$	$O[\mathbf{A_1}, S_1]$	$O[\mathbf{A_1}, S_2]$	$\ldots$	$O[\mathbf{A_1}, S_n]$
$\mathbf{A_2}$	$O[\mathbf{A_2}, S_1]$	$O[\mathbf{A_2}, S_2]$	$\ldots$	$O[\mathbf{A_2}, S_n]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{A_k}$	$O[\mathbf{A_k}, S_1]$	$O[\mathbf{A_k}, S_2]$	$\ldots$	$O[\mathbf{A_k}, S_n]$