More on Ordinals: ordinal arthimetic

April 1, 2016

Ordinals: review

This approach is due to von Neumann.

The word 'ordinal' comes from the Latin word *'ordinalis'* and originally referred to a book that described the order in which Church services were to be held.

Ordinals Ordinals represent *order types*: classes of isomorphic well-orderings. You can think of an ordinal as the paradigmatic exemplar of its order type.

ORDINAL HIERARCHY

Order Structure	Set Theoretic Representation	Name of Ordinal
	$\varnothing$	0_o
$\vert$	$\{0_o\} = \{\varnothing\}$	1_o
$\vert\vert$	$\{0_o, 1_o\} = \{\varnothing, \{\varnothing\}\}$	2_o
$\vert\vert\vert$	$\{0_o, 1_o, 2_o\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$	3_o
$\vdots$	$\vdots$	$\vdots$
$\vert\vert\vert\ldots$	$\{0_o, 1_o, 2_o, 3_o, \ldots\}$	ω
$\vert\vert\vert\ldots\vert$	$\{0_o, 1_o, 2_o, 3_o, \ldots, \omega\}$	$\omega + 1_o$
$\vdots$	$\vdots$	$\vdots$

A helpful way to think of the ordinal hierarchy is imagine it being built in stages.

> **Construction Principle**
>
> At each stage of the process, introduce a new ordinal which is the set of all ordinals that have been introduced at previous stages of the process.

Recall: a **well-ordering** is a linear ordering (i.e., an ordering that's *anti-symmetric*, *transitive*, and *total*) with the following property:

$$\forall R \ (R \subseteq S \rightarrow (\exists x \in R \ (\forall y \in R \ (x \leq y))))$$

Some Cool Facts about Ordinals:

- Every well-ordered set is order-isomorphic to some ordinal constructed in this way.
- $\in$ well-orders each ordinal: so each ordinal is an instance of the order type it represents.
- $\in$ *also* well-orders the set of all ordinals!

For ordinals α and β, we will say $\alpha <_o \beta$ iff $\alpha \in \beta$.

Ordinal Arithmetic

$(\alpha + \beta)$ represents the order type resulting from appending the type of ordering represented by β to one of the type represented by α.

Formally, where α' is the successor of α,

1. $\alpha +_o 0 = \alpha$
2. $\alpha +_o \beta' = (\alpha +_o \beta)'$
3. For limit ordinals λ, $\alpha + \lambda = \{x \in (\alpha +_o \beta) : \beta <_o \lambda\}$

This is an inductive definition of $+_o$. (I'm just going to write $+$ from here on out, though.)

Examples:

$2_o + 5_o$

$5_o + \omega$

$\omega + 5_o$

$\omega + \omega$

$(\omega + 1_o) + \omega$

$\omega + (\omega + 1_o)$

In some cases, $\alpha + \beta = \beta + \alpha$.
But not always. Give some examples where this succeeds vs fails:

In other words, ordinal arithmetic is *not commutative.* (**Bonus Question:** under what conditions does $\alpha + \beta = \beta + \alpha$?)

Ordinal Multiplication

$\alpha \times \beta$ represents the order type resulting from replacing each position in an ordering represented by β with an ordering of the type represented by α.

Formally,

1. $\alpha \times 0_o = 0_o$
2. $\alpha \times \beta' = (\alpha \times \beta) + \alpha$
3. for limit ordinals λ, $\alpha \times \lambda = \{x \in (\alpha \times \beta) : \beta <_o \lambda\}$

Examples:

$2_o \times 5_o$

$5_o \times \omega$

$\omega \times 5_o$

$\omega \times \omega$

$(\omega \times 2_o) + \omega$

$\omega \times (\omega + 2_o)$

$\omega \times (\omega \times (\omega + 1_o))$

When $+$ and $\times$ play together, just follow the order of operations dictated by the parentheses.

In some cases, $\alpha \times \beta = \beta \times \alpha$.
But not always. Give some examples where this succeeds vs fails:

In other words, ordinal multiplication is *not commutative.*

Big Numbers!

Ordinals provide a recipe for constructing large sets.

- A **successor ordinal** is an ordinal with an immediate predecessor. In other words, it is an ordinal α s.t. $\alpha = \beta + 1_o$ for some β. E.g. 5_o.
- A **limit ordinal** is an ordinal with no immediate predecessor. In other words, it is an ordinal λ such that $\beta + 1_o <_o \lambda$ for any β. E.g. ω.

Given an ordinal α, construct the set S_α as follows:

1. Consider each $\beta \leq_o \alpha$. Determine whether β is a limit or successor ordinal.
2. Then begin by lining up 0_o with the set of natural numbers.
3. Line up each successor ordinal with the result of applying the power set operation to whatever corresponds to the preceding ordinal.
4. Line up each limit ordinal (other than 0_o) with the union of everything that's been lined up with an ordinal so far.

We can use this procedure to pick out sets with larger and larger cardinalities. We can then use these large cardinalities to further extend our ordinal hierarchy. This, in turn, allows us to talk about sets with *even larger* cardinalities. So on and so forth.