The Bayesian Program

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The Three Parts of the Bayesian Program

- **Stage 1:** ***Precision.*** Individuals have real-valued degrees of belief.
- **Stage 2:** ***Synchronic Rationality (Probabilism).*** A rational agent's degrees of belief conform to the probability axioms.
- **Stage 3.** ***Diachronic Rationality (Conditionalization).*** The rational way to "update" on some information is to conditionalize on it.

Betting Rates

Unconditional Betting Rates. Let S be the stake, and let my betting rate for E be p.

Suppose we are betting on E. I bet $\$x$ on E; you bet $\$y$ against E.

- The *stake* is $\$(x+y)$.
- My *betting rate* on E is $\frac{x}{x+y}$
- Your *betting rate* on E is $\frac{y}{x+y}$

	Net Payoff for Bet on E	**Net Payoff for Bet against E**
E	$\$(1-p)\cdot S$	$-\$(1-p)\cdot S$
$\neg E$	$-\$p\cdot S$	$\$p\cdot S$

Personal Probabilities as Fair Betting Rates

What is your degree of belief in the proposition E? Which of the following do you prefer? A bet on E, or a bet against E?

$$\mathbf{B1} = \begin{cases} \$(1-p)\cdot 10 & \text{if } E \\ -\$p\cdot 10 & \text{o.w.} \end{cases} \qquad \mathbf{B2} = \begin{cases} \$p\cdot 10 & \text{if } \neg E \\ -\$(1-p)\cdot 10 & \text{o.w.} \end{cases}$$

Find the value of p such that you are *indifferent* between **B1** and **B2.** When you are indifferent between **B1** and **B2**, you are indifferent between betting on either side of E.

You are indifferent between X and Y just in case you don't prefer either one to the other.

Fair Betting Rate: Call this value of p your *fair betting rate.*

Conditional Betting Rates. Conditional bets are "called off" when the condition doesn't hold. Let your betting rate on A, given B, be p.

	Net Payoff for Bet on A, given B	**Net Payoff for Bet against A, given B**
$A\wedge B$	$\$(1-p)\cdot S$	$-\$(1-p)\cdot S$
$\neg A\wedge B$	$-\$p\cdot S$	$\$p\cdot S$
$\neg B$	$\$0$	$\$0$

When B is false, no one wins or loses the bet.

Synchronic Dutch Books

A set of betting rates is *coherent* if and only if it is not open to a sure-loss contract.

Stake = \$8. Your betting rate *on* B is $\frac{5}{8}$, and your betting rate *against* B is $\frac{6}{8}$.

EXAMPLE OF A SURE-LOSS CONTRACT:

	Net Payoff for Bet on B	**Net Payoff for Bet against** B	**Total net Payoff**
B	\$3	−\$6	−\$3
$\neg B$	−\$5	\$2	−\$3

Dutch Book Theorem: A set of fair betting rates is coherent if and only if they conform to the probability axioms.

1. **Normality.** The probability of any proposition X is somewhere between 0 and 1.

$$0 \leq \Pr(X) \leq 1 \tag{1}$$

   Trivial given the definition of betting rates.*

2. **Certainty.** Let Ω be a proposition that is certain to be true.

$$\Pr(\Omega) = 1 \tag{2}$$

   If your betting rate for Ω is $p < 1$, then you are asked to bet against Ω at a rate $(1 - p) > 0$, so will certainly lose $\$(1 - p) \cdot S$, because Ω is certain to happen.

3. **Additivity.** If propositions X and Y are *mutually exclusive*, then the probability of their disjunction is equal to the sum of their probabilities.

$$\text{If } X\&Y \text{ are mutually exclusive, } \Pr(X \vee Y) = \Pr(X) + \Pr(Y) \tag{3}$$

   Suppose that your betting rate on X is p, your betting rate Y is q, and your betting rate on $(X \vee Y)$ is $r \neq p + q$. (Suppose $r < p + q$). Consider the following three bets:

$$\mathbf{B_X} = \begin{cases} \$(1-p) & \text{if } X \\ -\$p & \text{o.w.} \end{cases}$$

$$\mathbf{B_Y} = \begin{cases} \$(1-q) & \text{if } Y \\ -\$q & \text{o.w.} \end{cases}$$

$$\mathbf{B_{X \vee Y}} = \begin{cases} -\$(1-r) & \text{if } X \vee Y \\ \$r & \text{o.w.} \end{cases}$$

	$\mathbf{B_X}$	$\mathbf{B_Y}$	$\mathbf{B_{X\vee Y}}$	**Total**
$X \wedge \neg Y$	$1-p$	$-q$	$-(1-r)$	$r-(p+q)$
$\neg X \wedge Y$	$-p$	$(1-q)$	$-(1-r)$	$r-(p+q)$
$\neg X \wedge \neg Y$	$-p$	$-q$	r	$r-(p+q)$

* Remember, your *betting rate* is how much you are willing to put into the pot divided by the total stake.
It is impossible to put in more than the stake. And it is impossible for your contribution to the stake, or the stake itself to be less than zero.

You'll be happy with each of these bets. Taking them all will result in a total of $r - (p+q)$, which is negative because $r < (p+q)$.
If $r > (p+q)$, then consider the following bets:

$$\mathbf{B_X} = \begin{cases} -\$(1-p) & \text{if } X \\ \$p & \text{o.w.} \end{cases}$$

$$\mathbf{B_Y} = \begin{cases} -\$(1-q) & \text{if } Y \\ \$q & \text{o.w.} \end{cases}$$

$$\mathbf{B_{X \vee Y}} = \begin{cases} \$(1-r) & \text{if } X \vee Y \\ -\$r & \text{o.w.} \end{cases}$$

Taking all three of the bets will result in a total of $(p+q) - r$, which is negative when $r > (p+q)$.

How to Learn?

Let $\Pr(X \mid Y)$ be the probability of X *conditional* on Y.
It is defined as follows:

$\Pr(X|Y)$ is, roughly, the probability that X is the case on the assumption that Y is the case.

$$\Pr(X \mid Y) = \frac{\Pr(X \wedge Y)}{\Pr(Y)} \tag{4}$$

Let H be some hypothesis. And let E be some evidence.

Bayes' Rule. Assume that $\Pr(E) > 0$. Then,

$$\Pr(H \mid E) = \frac{\Pr(E \mid H) \cdot \Pr(H)}{\Pr(E)} \tag{5}$$

$$= \frac{\Pr(E \mid H) \cdot \Pr(H)}{\Pr(E \mid H) \cdot \Pr(H) + \Pr(E \mid \neg H) \cdot \Pr(\neg H)} \tag{6}$$

This rule follows from the definition of conditional probability.

Proof. From the definition of conditional probability, we have that

$$\Pr(E \mid H) = \frac{\Pr(E \wedge H)}{\Pr(H)}$$

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So, $\Pr(E \mid H) \cdot \Pr(H) = \Pr(E \wedge H)$. And thus

$$\Pr(H \mid E) = \frac{\Pr(E \mid H) \cdot \Pr(H)}{\Pr(E)}$$

The third Stage of the Bayesian Program says the following:

CONDITIONALIZATION: If E is your total new evidence between time t_1 and time t_2, and you are *diachronically rational*, then, for every proposition P, your degree of belief in P at t_2 will be equal to your degree of belief in P, at time t_1, conditional on E.

$$\Pr_{t_2}(X) = \Pr_{t_1}(X \mid E)$$

The Commutativity of Evidence: The order in which you receive your evidence has no effect on your degrees of belief.