Risk-Taking and Tie-Breaking

Ryan Doody

1 Introduction

Instrumental rationality is about taking the best means to one's ends. According to the orthodoxy among decision theorists—expected utility theory (EUT)—taking the best means to one's ends involves two sorts of evaluations: first, you must determine what your ends are, and to what extent you value them (information that's encoded in an agent's utility function); second, you must determine how effective your means might be at realizing your ends (information that's encoded in an agent's credence function). The value of an option is its expected utility: roughly, the weighted average of how good or bad its potential outcome might be, where the weights correspond to the agent's credences in those outcomes resulting from its performance. According to orthodoxy, rational agents maximize expected utility.

Some philosophers—notably, Lara Buchak [e.g. Buchak, 2013]—argue that EUT is unduly restrictive: it doesn't appropriately take into account the agent's *attitude toward risk*. EUT, effectively, treats agents as if they were *risk-neutral* in virtue of the fact that it only takes *local* features of a gamble into account. But, argues Buchak, it's rationally permissible for agent's to care about a gamble's *global features*—the way its potential outcomes are situated in the space of possibilities (e.g., its minimum, its maximum, its variance, its spread, etc.)—in a way that EUT cannot properly accommodate.

In response, Buchak defends *risk-weighted expected utility theory* (REUT), which generalizes EUT by adding a third parameter into the evaluation of a gamble's subjective value. In addition to a utility function (representing how the agent values her ends) and a probability function (representing what the agent thinks might happen), REUT represents an agent's attitude toward risk by attributing to her a *risk function*, r. Unlike EUT, which weights the value of each potential outcome by that outcome's probability, REUT weights the value of each potential outcome by a function r of the probability of getting something at least as valuable. If r is convex, the value of outcomes above the minimum will contribute less to the overall value of the gamble. Agents with convex risk functions are, using Buchak's terminology, *risk-avoidant*: the value they will assign to a risky gamble will be lower than its expected value.

Although Buchak may well be right that EUT is inadequate, I will argue that REUT has some rather unpalatable consequences of its own. First, it's not obvious that it correctly represents what it is to be risk-averse: REUT will, I argue, sometimes undervalue gambles that it shouldn't. Second, and relatedly, REUT has some counterintuitive consequences regarding tie-breaking. When you're indifferent between two options, it's rationally permissible to take either. One way to decide between two indifferent options is to flip a fair coin, taking the one if it lands heads and the other if it lands tails. Offhand, it seems rationally permissible to employ such a tie-breaking procedure. However, if you are risk-avoidant, there will be cases in which the value of deciding by coin-flip will be lower than the value of choosing one of the options outright. And so, in such cases, REUT says that it is *not* permissible to employ a tie-breaking procedure.

In the next section, I will present REUT using a couple of examples. In section 3, I will argue that these examples present a challenge for Buchak's claim that REUT faithfully captures what it is to be risk-averse. In section 4, I will argue that REUT offers counterintuitive guidance in cases of tie-breaking.

2 Risk-weighted Expected Utility Theory

The best way to understand REUT is to contrast it with EUT. Let's look at what EUT says in more detail.

Let $L = \{x_1, E_1; x_2, E_2; \ldots x_n, E_n\}$ be a gamble, where $u(x_1) \leq u(x_2) \leq \cdots \leq u(x_n)$.

Expected Utility

$$EU(L) = \sum_{i=1}^{n} Cr(E_i) \cdot u(x_i)$$
$$= u(x_1) + \left(\sum_{i=2}^{n} Cr(E_i)\right) \cdot (u(x_2) - u(x_1)) + \cdots + Cr(E_n) \cdot (u(x_n) - u(x_{n-1}))$$

EUT says that you ought to maximize expected utility. The expected utility of a gamble is the probability-weighted average of the values of its potential outcomes. Here's one way to calculate the expected utility of a gamble. First, weight the value of each of its potential outcomes by the probability of that outcome occurring. Then, take the sum of those probability-weighted values.

Here's a different, equally as accurate, way of calculating a gamble's expected utility. First, order the gamble's potential outcomes from worst (least preferred) to best (most preferred). The expected value of the gamble is *at least* as high as the value of its worst potential outcome. So, start with the value of the worst potential outcome. Then add to this minimum value the difference between it and the next highest potential value (i.e, the second-worst value), weighted by the probability of getting at least that amount. Then add to *that* the difference between the second-worst value and the next highest potential value (i.e., the third-worst value), weighted by the probability of getting something at least as valuable as it. And so on and so forth, until we reach the best potential outcome.

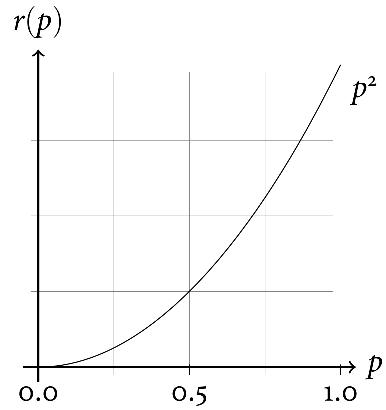
When there are only two outcomes, it's easy to see that these two methods coincide. Let x_1 and x_2 be the worst and best outcomes, respectively. And let p be the probability of x_2 occurring. The expected utility of such a gamble is: $p \cdot u(x_2) + (1 - p) \cdot u(x_1)$. This expression can be rewritten as: $u(x_1) + p \cdot (u(x_2) - u(x_1))$, which is the minimum value of the gamble plus the amount you might gain above that minimum weighted by the probability of realizing that gain.

The *risk-weighted* expected utility of a gamble can be calculated in an analogous way—with one crucial difference: instead of weighting the potential gains by their probabilities, REUT weights these potential gains by a *function* of their probabilities. So, to use the example from the previous paragraph, the risk-weighted expected utility of the gamble would be: $u(x_1) + r(p) \cdot (u(x_2) - u(x_1))$. If r is convex, $r(p) < p$ for all non-extremal values. And so, the amount you might gain above the minimum will contribute less to the overall instrumental value of the gamble than it does on EUT.

Risk-Weighted Expected Utility

$$REU(L) = u(x_1) + r\left(\sum_{i=2}^{n} Cr(E_i)\right) \cdot (u(x_2) - u(x_1)) + \cdots + r\left(Cr(E_n)\right) \cdot (u(x_n) - u(x_{n-1}))$$
$$= \sum_{j=1}^{n}\left(r\left(\sum_{i=j}^{n} Cr(E_i)\right) \cdot \left(u(x_j) - u(x_{j-1})\right)\right)$$

REUT is a *generalization* of EUT: the two views coincide when $r(p) = p$, for all probabilities p. The risk function is subject to the following constraints: for all p, $0 \leq r(p) \leq 1$; $r(0) = 0$ and $r(1) = 1$; r is non-decreasing. For the sake of concreteness, let's look at a specific convex risk function: $r(p) = p^2$. (This is Buchak's go-to example of a risk function characterizing risk-aversion.)



The risk-function "measures how an agent structures the potential realization of some of his aims," [Buchak, 2013, p. 54]. In order to better see how agents with the risk function $r(p) = p^2$ structure the potential realization of their aims, let's look at a couple of examples.

You're at the racetrack placing bets on the horses. You are considering whether to bet *for* Easy Street (f), which pays out 4 utils if Easy Street wins (E) and 2 utils if she doesn't; or to bet '*gainst* Easy Street (g), which pays out 3 utils if Easy Street doesn't win ($\neg E$) and 1 util if she does. Let's suppose that your credence in E is .25, your credence in $\neg E$ is .75, and your risk function is $r(p) = p^2$.

	E (1/4)	$\neg E$ (3/4)
f	4	2
g	1	3

First, note that both gambles have the same expected utility: 2½.

$$\begin{aligned} EU(f) &= 2 + \frac{1}{4} \cdot (4 - 2) \\ &= 2 + \frac{1}{4} \cdot 2 = 2^1/_2 \end{aligned}$$

$$\begin{aligned} EU(g) &= 1 + \frac{3}{4} \cdot (3 - 1) \\ &= 1 + \frac{3}{4} \cdot 2 = 2^1/_2 \end{aligned}$$

Interestingly, when $r(p) = p^2$, both gambles also have the same *risk-weighted* expected utility: 2⅛. Because $r(p) = p^2 \leq p$ (representing someone who is risk-avoidant), as we should expect, these gambles have lower REU than EU.

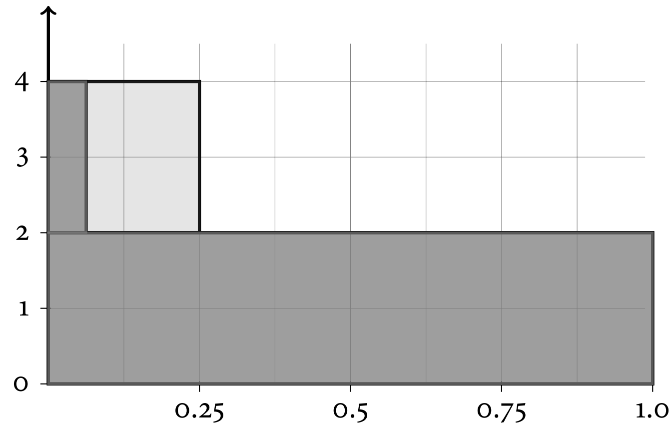


Figure 1: The risk-weighted expected utility of gamble f (2.125).

$$\begin{aligned} REU(f) &= 2 + \left(\frac{1}{4}\right)^2 \cdot (4 - 2) \\ &= 2 + \frac{1}{16} \cdot 2 = 2^1/_8 = 2.125 \end{aligned}$$

$$\begin{aligned} REU(g) &= 1 + \left(\frac{3}{4}\right)^2 \cdot (3 - 1) \\ &= 1 + \frac{9}{16} \cdot 2 = 2^1/_8 = 2.125 \end{aligned}$$

If you take gamble f, you are guaranteed to get something at least as valuable as 2 utils, and you have a 25% chance of getting something 2 utils of value more valuable. If you're risk-avoidant, the potential improvements above the guaranteed minimum are "discounted", contributing less to the overall instrumental value of the gamble than it does relative to EUT (see Figure 1). For EUT, the 25% chance of improving above the minimum adds .5 units of value; for REUT, it only adds .125 units of value. (In Figure 1, each square represents .125 utils of value. The lighter gray represents the gamble's EU and the darker gray represents its REU. For EUT, the chance of improvement above the minimum is worth four .125-util-squares, which totals .5 utils. For REUT, however, it's only worth two half .125-util-squares, which totals .125 utils.)

If you take gamble g, however, you are only guaranteed to get something at least as valuable as 1 util, but you have a larger chance—75% rather than 25%—of getting something 2 utils of value more valuable than that minimum. Again, if you're risk-avoidant, this potential improvement contributes less to the overall instrumental value of the gamble than it does relative to EUT (see Figure 3). For

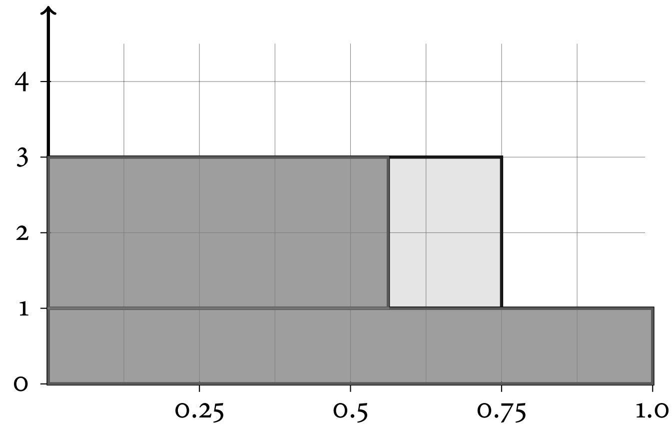


Figure 2: The risk-weighted expected utility of gamble *g* (2.125).

EUT, the 75% chance of improving above the minimum adds 1.5 units of value; for REUT, it only adds 1.125 units of value. (As Figure 2 shows, for EUT, the chance of improvement above the minimum is worth twelve .125-util-squares, totaling 1.5 utils. And, for REUT, that same chance of improvement is only worth eight full .125-util-squares and two half .125-util squares, totaling 1.125 utils.)

Although gamble *f* has a higher minimum than gamble *g*, gamble *g* affords a greater chance of improvement beyond that minimum than gamble *f* does. Somewhat surprisingly, according to *both* EUT and REUT, these two features balance out, making the two gambles equally valuable. Both views agree that you should be indifferent between betting for and betting 'gainst Easy Street. While the two views agree that gamble *g* and gamble *f* have the same value, they disagree about what that value is. EUT values the two gambles at 2.5, while REUT values them at 2.125.

Suppose that Eunice is an expected utility maximizer and her sister, Reu, is a risk-weighted expected utility maximizer (with the risk function $r(p) = p^2$). Suppose they share the same values and beliefs. (Also, for the sake of notational simplicity, let's suppose that they value money linearly: $u(\$x) = x$.) Euince will be indifferent between gamble *f*, gamble *g*, and \$2.50. Reu, on the other hand, will be indifferent between gamble *f*, gamble *g*, and \$2.125. Let's say that if *S* is indifferent between some gamble and \$*x*, \$*x* is that gamble's *sure-thing cash equivalent* for *S*. \$2.50 is gamble *f*'s and *g*'s sure-thing cash equivalent for Euince; \$2.125 is gamble *f*'s and *g*'s sure-thing cash equivalent for Reu. Eunice would be willing to pay more to buy of the two gambles than Reu would.

But, instead of paying the bookie for one of the gambles, suppose that Euince and Reu are offered the opportunity to select one of them for free. What, respectively, should they do? They are both indifferent between the two, so it's permissible for them to choose either. When they're indifferent between two options, the sisters typically decide by flipping a fair-coin: if the coin lands heads, take gamble f; if it lands tails, take gamble g. Call this a *tie-breaking procedure*. Is it rationally permissible to employ a tie-breaking procedure in this case? It depends on the sister. For Euince, it is. Doing so has the same expected utility as selecting either of the two gambles outright. For Reu, on the other hand, it is *not*. The 50/50 "mixture" of the two gambles has lower risk-weighted expected utility than selecting one of the gambles outright.

In the next section, we'll take a closer look at why the 50/50 "mixture" of the two gambles—let's call it $f \oplus_{1/2} g$—has lower risk-weighted expected utility than either of the two gambles themselves. And I'll argue that this is a counterintuitive consequence of REUT; one that raises questions about the view's success in accurately characterizing what it is to be risk-averse.

3 Mean, Variance, and Risk-aversion

Reu, like Euince, is indifferent between gamble f and gamble g. Unlike Euince, it's not permissible for Reu to employ a 50/50 tie-breaking procedure—like, using the flip of a fair-coin—to make her selection.

Decide by Coin Flip

	Heads		Tails	
	E	$\neg E$	E	$\neg E$
$f \oplus_{1/2} g$	4	2	1	3

Employing such a procedure, which corresponds to a 50/50 "mixture" between the two gambles, has lower risk-weighted expected utility than either of the gambles themselves. This can be demonstrated by doing the calculation:

$$
\begin{aligned}
REU\left(f \oplus_{1/2} g\right) &= 1 + \left(\frac{7}{8}\right)^2 \cdot (2-1) + \left(\frac{4}{8}\right)^2 \cdot (3-2) + \left(\frac{1}{8}\right)^2 \cdot (4-3) \\
&= 1 + \frac{49}{64} + \frac{16}{64} + \frac{1}{64} = 2^{1}/_{32} = 2.03125
\end{aligned}
$$

This might seem, however, to be somewhat counterintuitive. Because f and g have the same expected value (or, *mean*), their probabilistic "mixture" does as well. If their mixture is less valuable, then, it might seem like this must be because it is more risky than either of the gambles considered alone. But how would "mixing" together the two gambles introduce more risk? If anything, it seems like "mixing" the two gambles together creates something *less* risky![1]

[1] Consider, for example, the oft-touted financial advice to *diversify* one's portfolio—the practice of

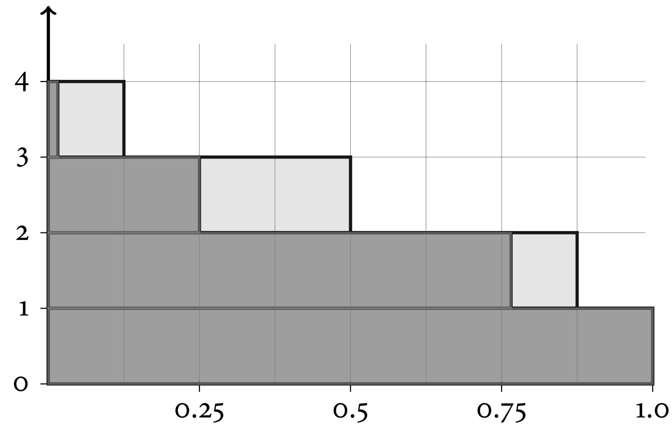


Figure 3: The risk-weighted expected utility of a fifty-fifty lottery between gamble *f* and gamble *g* (2.03125).

It might be helpful, then, in the service of evaluating REUT, to say more about what risk *is* and how it can be measured. One natural way of measuring risk is with *variance*. In general, the variance of a distribution measures the extent to which its values deviate from the average. The variance of a gamble, in particular, measures how far away from its expected value the value of its potential outcomes are. There is a sense, then, in which the higher the variance, the riskier the gamble.[2]

Let $L = \{x_1, E_1; x_2, E_2; \ldots x_n, E_n\}$ be a gamble, and let $EU(L)$ be its expected value. Then, the variance of L is:

$$\textsc{Var}(L) = \sum_{i=1}^{n} Cr\left(E_i\right) \cdot \left(u(x_i) - EU(L)\right)^2$$

Roughly, the variance of a gamble is the probabilistically-weighted average of the "distance" between each of its potential outcomes and its expected value.

Is $f \oplus_{1/2} g$ riskier—in the sense of having a higher *variance*—than gamble f and gamble g? The answer is: no. In addition to having the same mean, the three also have the same variance: 0.75. If the risk of a gamble is given by its variance, the three are all equally risky. What, then, justifies Reu in assigning a lower instrumental value to the one than to the other two?

A gamble's variance, however, is not the only way of measuring its riskiness.

spreading one's money among various different investments—in order to reduce risk. Investments, in virtue of paying out sums of money with various probabilities, are gambles. Diversifying, then, is akin to taking a "mixture" of these gambles.

[2] Modern Portfolio Theory [Markowitz, 1952] primarily uses variance (and its close cousin, standard deviation) as a measure of risk.

Instead, we could compare the riskiness of two gambles by checking to see if the one is a *mean-preserving spread* of the other. In fact, this is how Buchak defines what it is to be risk-averse. She says,

> [A]n agent is *generally risk-averse in money* (or any quantitative good) just in case of any two gambles f and g such that g can be obtained from f by a mean-preserving spread, the agent weakly prefers f; that is, of any two gambles with the same average monetary value, the agent weakly prefers the one that is less spread out, if such a judgment can be made," [Buchak, 2013, p. 21-2].

Roughly, B is a mean-preserving spread of A if, they have the same mean, and B can be obtained from A by adding "noise". Impressionistically, mean-preserving spreads shift value away from the center of a distribution out toward its tails.[3] If one gamble is a mean-preserving spread of another, it's riskier.

However, the relation "is a mean-preserving spread of" induces only a partial ordering on gambles. Obviously, if two gambles have different means, neither can be a mean-preserving spread of the other. Yet, intuitively, one might be more risky than the other. But even among gambles with the same mean, it might be that neither is a mean-preserving spread of the other. Our three gambles—f, g, and $f \oplus_{1/2} g$—are a case in point. If B is a mean-preserving spread of A, B will have a higher variance than A. But our three gambles all have the same mean and the same variance, so none are mean-preserving spreads of the other. The lower instrumental value that Reu assigns to $f \oplus_{1/2} g$ cannot be because it is riskier in the sense of being a mean-preserving spread of f or g.

What other features are there that could justify assigning a lower instrumental value to $f \oplus_{1/2} g$ than to f and to g? One possible difference is that, whereas f and g both have two potential outcomes, the 50/50 lottery between them has four. Another possible difference—related to the first—is that, while the difference in value between f's and g's best and worst outcomes is only 2, the difference in value between their mixture's best and worst outcomes is 3. But it's at least not obvious how these features, in general, relate to a gamble's risk. Having more potential outcomes, for example, is surely compatible with being *less*, rather than more, risky.

[3] Mean-preserving spreads were first explored in Rothschild and Stiglitz [1970], whose major contribution involved proving that the following three properties are equivalent: (1) $EU(X) \succeq EU(Y)$, for all concave utility-functions; (2) Y has more weight in its tails than X; (3) The outcomes of Y are distributed just like X's plus noise.

4 Tie-breaking

Let's return to our story of Eunice and Reu. They are both indifferent between gamble f (betting *for* Easy Street) and gamble g (betting *'gainst* Easy Street), and must decide on which to select. For Eunice, it's permissible to employ a tie-breaking procedure, like flipping a fair coin, in making her selection. For Reu, however, it is not.

Because Rue is indifferent between f and g, her reasons for taking the one are perfectly in balance with her reasons for taking the other. She has no rational basis for *choosing* the one over the other. This is—to borrow a distinction from Ullmann-Margalit & Morgenbesser (1977)—a situation that calls for *picking* rather than *choosing*. When we choose one option over another, we do so on the basis of the reasons we have that favor the one over the other. One way to think about picking, on the other hand, is that "when we are in a genuine picking situation we are in a sense transformed into a chance device that functions at random and effects arbitrary selections [...]" [Ullmann-Margalit and Morgenbesser, 1977, p. 773].

Suppose that Reu knows that, later, she will be in a picking situation: she'll have to select between options that she is (and knows she will continue to be) indifferent between. How should Reu value ending up in such a situation?

More concretely, let's suppose that Reu is in a hallway facing two doors. She knows that behind Door #1, there is a table on which sits three prizes: a ticket for gamble f, a ticket for gamble g, and $\$f$ ($= \$g$), where $\$f$ ($\$g$) is gamble f's (g's) sure-thing cash equivalent. If she opens Door #1, she gets to pick one of those three prizes. On the other hand, she knows that behind Door #2, there is a table with one prize on it: $\$2^1/_{16}$. If she opens Door #2, that's what she'll get.

It's (mostly) clear what Eunice would do in this situation: she would, first, open Door #1 and then select one of the three options (perhaps with the aid of some tie-breaking procedure). Even though Eunice might not know which prize she would select were she to open Door #1, doing so has greater value for her than opening Door #2. This is because, no matter how Eunice thinks about going on to pick between the three prizes if she opens Door #1, the value she assigns to opening Door #1—its expected value—is equal to the value she assigns to each of the three prizes behind it. And that value is greater than that of the $\$2^1/_{16}$ she is sure to get by opening Door #2 instead.

Opening Door #1 is a sensible thing for Eunice to do because—while opting for Door #1 doesn't *guarantee* a better outcome than opting for Door #2 would—she prefers each of the prizes behind Door #1 to the one behind Door #2. This is an example of Eunice obeying the following principle:

Menu Superiority

> If you know that φing will present you with a menu of options such that, for each of the items on that menu, you prefer it to the items on the menu you'd be presented with if you didn't φ, then you rationally ought to prefer φing.

For example, when choosing between two restaurants, if you know that you prefer each of the dishes on offer at the first to all the dishes on offer at the second, it would be irrational to prefer going to the second. Eunice obeys this principle.

What about her sister, Reu? Does *she* obey the principle? Which does she prefer: Door #1 or Door #2? As I'll demonstrate in this section, it's not straightforward. Let's explore some of the possible, conflicting ways of approaching the question.

(1) Pick One of the Three Outright. Here is one way of thinking about Reu's situation. Because Reu is indifferent between the three options behind Door #1, for all she knows, she might pick any of the three. She has no reason to think she's more likely to pick any one of them than the others, so she should assign equal credence to each of the three possibilities.[4]

Gamble Between f and g and $f

	Pick f		Pick g		Pick $\$f$	
	E	$\neg E$	E	$\neg E$	E	$\neg E$
$\oplus_{1/3}(f, g, \$f)$	4	2	1	3	$2^1/8$	$2^1/8$

$$\begin{aligned} REU\left(\oplus_{1/3}(f, g, \$f)\right) =& 1 + \left(\frac{11}{12}\right)^2 \cdot (2-1) + \left(\frac{8}{12}\right)^2 \cdot (2^1/8 - 2) \\ &+ \left(\frac{4}{12}\right)^2 \cdot (3 - 2^1/8) + \left(\frac{1}{12}\right)^2 \cdot (4-3) \\ =& 1 + \frac{121}{144} + \frac{8}{144} + \frac{14}{144} + \frac{1}{144} = 2 \end{aligned}$$

If Reu thinks that, were she to open Door #1, it's equally likely that she'd select any one of the three prizes over the others, then opening Door #1 is like a lottery with a $1/3$ chance of paying out gamble f, a $1/3$ chance of paying out gamble g, and a $1/3$ chance of paying out $\$f$ (which, for Reu, is $\$2^1/8$). Given Reu's attitude toward risk, such a lottery is valued at \$2. Because opening Door #2 will result in $\$2^1/16$

[4] The reasoning here appears to appeal to something like the *Principle of Indifference*, which is controversial. I don't think the argument turns on the truth of this principle in its full generality, however. For my purposes, it's enough that it be reasonable for someone like Reu to assign credences to her future actions in the way described above. It needn't be the case that she *must*—on the pains of irrationality—do so, only that this is an epistemically reasonable reaction to her situation.

for sure, doing so has higher value for her than opening Door #1. So, if this is the procedure that Reu knows she'll use when picking between the three prizes behind Door #1, she should open Door #2 instead.

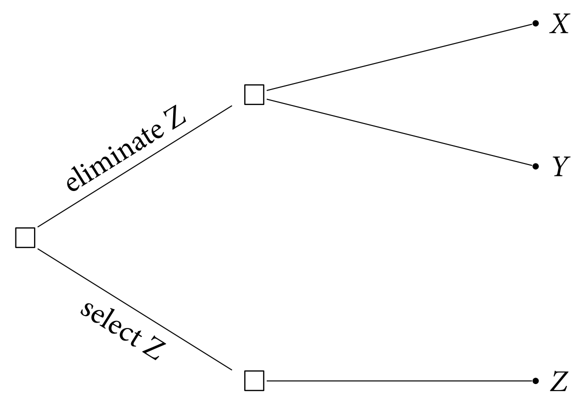
But, by preferring Door #2 to Door #1, Reu violates *Menu Superiority*: she knows, for each of the prizes behind Door #1, that she prefers it to the prize behind Door #2 and yet prefers opening Door #2. In fact, given that one of the things she could do is to open Door #1 and then select the sure-thing $2 1/8, there is something Reu could do that would *guarantee* a better outcome than the one she knows she'll bring about by opening Door #2. And yet—if it's reasonable for Reu to think the she will employ such a procedure when selecting between the prizes behind Door #1—she should nevertheless prefer to open Door #2.

That might seem absurd. But there are other ways we might approach Reu's situation—ways which might prove to be less absurd. Let's take a look.

(2) Use a Sequential Pairwise Procedure. Alternatively, when picking between the three prizes behind Door #1, Reu might employ a procedure that involves evaluating the options pair-by-pair. There are two such procedures: the first, let's call *Process of Elimination*; and the second, let's call *Tournament Method.* Let's look at each in turn.

- *Process of Elimination.* When deciding between three options, $\{X, Y, Z\}$, you first decide whether to select one of the options outright or to eliminate it from the running; if you opt to eliminate it, then you go on to decide between the remaining two.

Process of Elimination (Z)

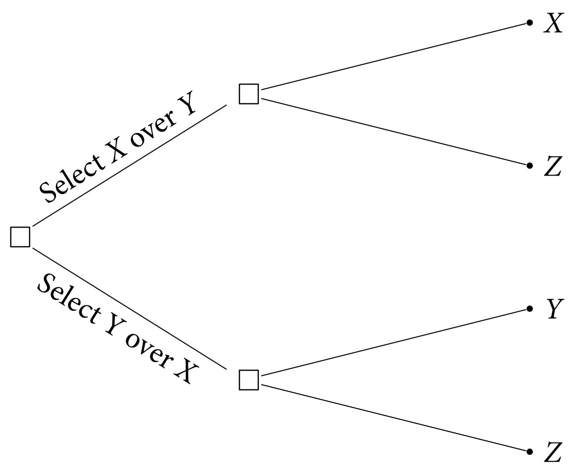


Because there are three ways to choose one from a set of three, there are three different ways of employing this procedure: you can first decide whether to select or eliminate X, choosing between Y and Z if you opt for elimination (call this *Eliminate X*); or you can first decide whether to select or eliminate

Y, choosing between *Z* and *X* if you opt for elimination (*Eliminate Y*); or, lastly, you can decide whether to select or eliminate *Z*, choosing between *X* or *Y* if you opt for elimination (*Eliminate Z*). Let's call these *ways of setting the agenda*. In this case, there are three ways of setting the agenda.

- *Tournament Method.* When deciding between three options, $\{X, Y, Z\}$, you first decide which out of two of them will proceed to the next round; then, at the next round, the "winner" of the first round faces-off against the remaining option.

  Tournament Method (Z Gets a "Bye")

  Again, because there are three options, there are three different ways of employing this procedure: one where *X* gets a bye, one where *Y* gets a bye, and one where *Z* gets a bye. Here, too, there are three ways of setting the agenda.

We have two different sequential pairwise procedures and, for each, three ways of setting the agenda. When picking between the three prizes between Door #1, does it matter which procedure is used and how the agenda is set? Should it? If you're Eunice, it doesn't matter. No matter which procedure she thinks she might implement and however she might set the agenda were she to open Door #1, opening Door #1 will have greater expected value than opening Door #2. Furthermore, because Eunice is indifferent between the three prizes behind Door #1, none of the procedures—and none of the ways of setting the agenda—make it any more or less likely that she'll end up with one of the prizes rather than another. Consequently, she will have no reason to favor adopting any one of these procedures over any other. Just as she is indifferent between the results of the procedure—that is, the three prizes on the table—she is likewise indifferent between the procedures themselves. Eunice obeys the following principle:

Agenda Invariance

> When deciding from a menu of options, how you set the agenda should have no effect on what it's rational to do.[5]

When choosing an entree from a menu, it shouldn't matter whether you start at the top or the bottom of the list. If it would be irrational of you to choose the Jellied Eel were you to start at the top, it should be irrational for you to choose the Jellied Eel were you to start from the bottom instead. How the decision is "framed" shouldn't matter.[6]

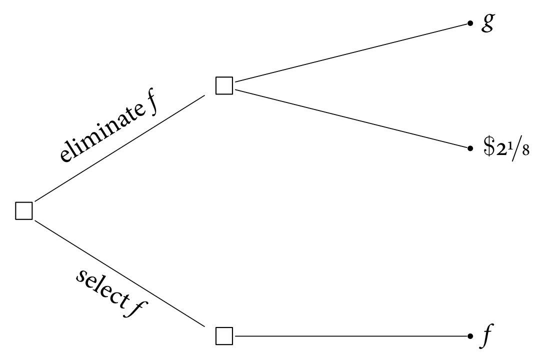
But as I'll demonstrate, if you're Reu, it *does* matter. Reu, unlike her sister, violates *Agenda Invariance*: for her, how the agenda is set does effect what it's rational for her to do. Let's consider each of the two sequential pairwise procedures outlines above, in turn.

- **Process of Elimination.** As mentioned before, when there are three options on the menu, there are three ways of setting the agenda. In Reu's case, the menu behind Door #1 includes the ticket for gamble f, the ticket for gamble g, and $\$f$ ($= \$2^1/_8$)—the sure-thing cash equivalent. Thus, there are a three corresponding ways of setting the agenda when using *Process of Elimination*: she can decide whether or not to eliminate f, or she can decide whether or not to eliminate g, or she can decide whether or not to eliminate $\$f$. Let's explore each of these.

  Eliminate f. On this way of setting the agenda, Reu first decides whether to select or to eliminate gamble f. If she opts to eliminate f, she's then faced with a choice between gamble g and $\$2^1/_8$—its sure-thing cash equivalent.

[5] More precisely: When deciding from a menu M of options, if there is a way of setting the agenda $A(M)$ such that, were you to set the agenda that way, rational choice would result in the selection of $X \in M$, then *every* way of setting the agenda is likewise such that, were you to use *it*, rational choice would result in the selection of X. What it's rational for you to do is *invariant* across agendas.

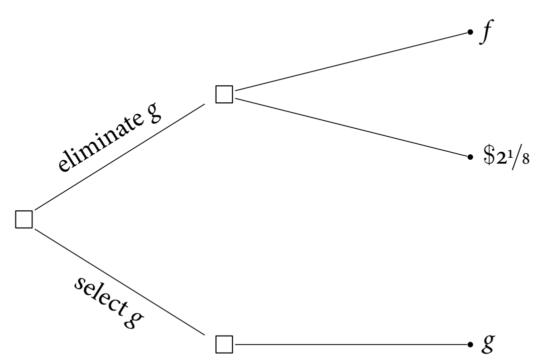
[6] In a series of influential papers [Tversky and Khaneman, 1979, 1984, 1986], Daniel Khaneman and Amos Tversky argue that, for many of us, how a decision-problem is framed *does* matter. However, their examples pertain to how particular *options* are framed—that is, how various features of an option are described—and not to the way in which those options are considered. Holding fixed how the options themselves are described, should it matter e.g. the order in which they are considered? Furthermore, Khaneman and Tversky are engaged in a description project, not a normative one. They agree that principles, like the one above, are "normatively essential" [Tversky and Khaneman, 1986].



What should Reu do? She knows that if she elects to *eliminate f*, she will face a choice between two prizes—gamble *g* and \$2¹/₈—between which she is indifferent. Because she is indifferent between the two, for all she knows, she might pick either when facing such a choice. So, she assigns equal credence to ending up with either of the prizes in the event that she eliminates *f*. Thus, *eliminate f* is like a lottery with a ½ chance of paying out gamble *g* and a ½ chance of paying out \$2¹/₈. Given Reu's attitude toward risk, such a lottery is valued at \$1⁶³/₆₄.[7] Because gamble *f*—which is obviously what will result were Reu to elect to *select f* rather than to eliminate it—is valued at \$2¹/₈, Reu should prefer selecting gamble *f* over eliminating it.

Result: If this is the way the agenda is set, Reu will select gamble *f*.

Eliminate g. On this way of setting the agenda, Reu first decides whether to select or to eliminate gamble *g*. If she opts to eliminate *g*, she's then faced with a choice between gamble *f* and \$2¹/₈—its sure-thing cash equivalent.



Because Reu is indifferent between gamble *f* and \$2¹/₈, the option to *eliminate g* is like a 50/50 lottery between the two, which is valued at \$2⁵/₆₄ given her attitude toward risk.[8] Because gamble *g*—which is

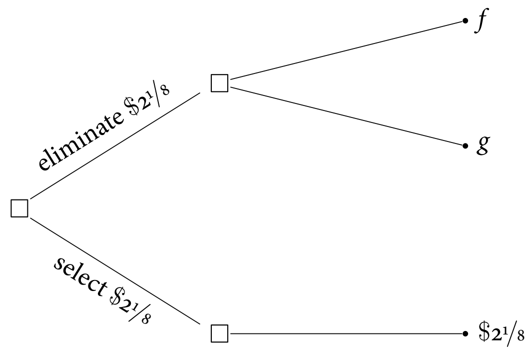
[7] This is not obvious. See appendix A for details of the calculation.

[8] Again, see appendix A for the calculations.

what will result were she to *select g* rather than eliminate it—is valued at $\$2^1/_8$, Reu should prefer selecting gamble *g* over eliminating it.

Result: If this is the way the agenda is set, Reu will select gamble *g*.

Eliminate $\$2^1/_8$. On this way of setting the agenda, Reu decides whether to select or to eliminate $\$2^1/_8$. If she opts to eliminate it, she goes on to face a choice between the two gambles.



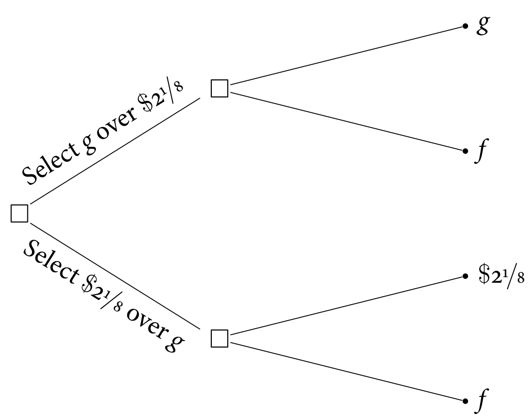
Because Reu is indifferent between the two gambles, choosing between them is like a 50/50 lottery. As we saw in Section 3, a 50/50 lottery between gamble *f* and gamble *g* is valued at $\$2^1/_{32}$. This is worth less than the $\$2^1/_8$ Reu will get by selecting it outright. So, that's what she should do.

Result: If this is the way the agenda is set, Reu will select the $\$2^1/_8$.

There are three things to note. First, if Reu uses *Process of Elimination* to select between the three prizes behind Door #1, her indifference among the three is recapitulated in her decision about how to set the agenda. Each way of setting the agenda results in each one of the prizes. Second, because each of the three prizes is the result of one of the ways of setting the agenda, Reu violates *Agenda Invariance*. Different ways of setting the agenda result in different prizes, so the agenda *does* have an effect on what it's (seemingly) rational for Reu to do. Finally—returning to Reu's choice between Door #1 and Door #2—if she knows that, were she to open Door #1, she'd go on to select between the three prizes using *Process of Elimination*, then, because each way of setting the agenda will result in a prize that is valued at $\$2^1/_8$ and because behind Door #2 there is only $\$2^1/_{16}$, Reu should prefer Door #1 to Door #2. And this reveals a more serious violation of *Agenda Invariance*—for, as we saw above, if Reu thinks that she'd Pick One of the Three Outright instead, she'd prefer Door #2. For Reu, the agenda matters.

- **Tournament Method.** With this method, as well, there are three different ways of setting the agenda. Reu can decide to allow gamble *f* to get a bye into the next round, or to allow gamble *g* to get a bye into the next round, or to allow the $2^1/_8$ sure-thing cash equivalent to get a bye into the next round. Let's look at each.

f gets a bye. On this way of setting the agenda, Reu first decides whether or not to select gamble *g* over $2^1/_8$. Then, whichever is selected is put in contention with gamble *f*. And Reu decides which of those two to take home.

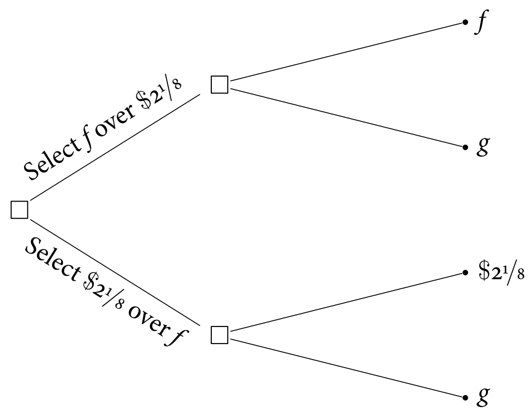


Reu knows that if she selects gamble *g* over $2^1/_8$, she'll then face a choice between gamble *g* and gamble *f*. Because she is indifferent between the two, for all she knows, she might pick either one. So, selecting gamble *g* over $2^1/_8$ is like a 50/50 lottery between gamble *g* and gamble *f*. Given her attitude toward risk, Reu values such a lottery at $2^1/_{32}$. On the other hand, if she selects $2^1/_8$ over gamble *g*, she'll then face a choice between $2^1/_8$ and gamble *f*. For similar reasons, she can think of this choice as akin to a 50/50 lottery between the two. Given her attitude toward risk, she values it at $2^5/_{64}$.[9] The latter is better than the former, so Reu should opt to select $2^1/_8$ over gamble *g*.

Result: If this is the way the agenda is set, Reu will take home either $2^1/_8$ or gamble *f*. (This way of setting the agenda is worth $2^5/_{64}$ to her.)

g gets a bye. On this way of setting the agenda, Reu decides whether or not to select gamble *f* over $2^1/_8$. Whichever wins goes on to face gamble *g* in the final round.

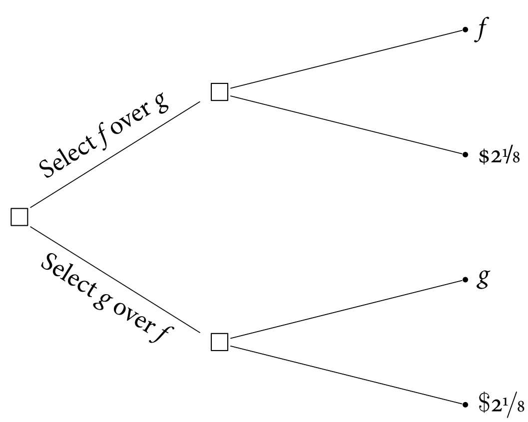
[9] See appendix A for details.



For reasons analogous to those in the previous example, we can represent Reu's choice as between two 50/50 lotteries—one between gamble *f* and gamble *g*, and the other between \$2¹/₈ and gamble *g*. The former, as we saw, is valued at \$2¹/₃₂, while the latter is valued at \$1⁶³/₆₄.[10] The former is better than the latter, so Reu should opt to select gamble *f* over \$2¹/₈.

Result: If this is the way the agenda is set, Reu will take home either gamble *f* or gamble *g*. (This way of setting the agenda is worth \$2¹/₃₂ to her.)

\$2¹/₈ **gets a bye.** Lastly, on this way of setting the agenda, Reu first decides whether or not to select gamble *f* over *g*. The winner faces off against \$2¹/₈—their sure-thing cash equivalent.



Like above, we can compare the 50/50 lottery between gamble *f* and \$2¹/₈ with the 50/50 lottery between gamble *g* and \$2¹/₈. The former is valued at \$2⁵/₆₄, while the latter is valued at \$1⁶³/₆₄. The former is better than the latter, so Reu should opt to select gamble *f* over *g*.

[10] See appendix A for details.

> *Result:* If this is the way the agenda is set, Reu will take home either gamble *f* or \$2 1/8. (This way of setting the agenda is worth \$2 5/64 to her.)

> Again, there are three things to note. First, unlike with *Process of Elimination*, Reu values these different ways of setting the agenda differently—she disprefers the agenda in which gamble *g* gets a bye to the other two. She values picking from the menu $\{f, \$2\frac{1}{8}\}$ over picking from the menu $\{f, g\}$. Setting the agenda in a way that she prefers, though, effectively removes gamble *g* from the running. (This is odd because, given Reu's indifference between gamble *g* and the other prizes, it seems like it very much *is* still in the running.) Second, we have yet another example of Reu violating *Agenda Invariance.* Not only might different agendas result in different prizes, this is a case in which Reu has a clear preference for some of the ways of setting the agenda over others. And, lastly, if Reu knew that, were she to open Door #1, she'd select between the prizes using the *Tournament Method*, then, because she prospectively values deciding in this way at \$2 5/64, she'd prefer to open Door #1 over Door #2.

Should Reu open Door #1 or Door #2? It seems like it depends on how she believes she might go about deciding between the prizes between Door #1. If she thinks she'll implement *Process of Elimination* or *Tournament Method*, then she should open Door #1. (Which prize she'll end up with will depend on how she sets the agenda.) If, on the other hand, she thinks she'll just pick one of three outright, then she should open Door #2 instead. How she thinks her future decision—one that only concerns prizes that she is indifferent between—will be structured makes a (some might say, "implausible") difference to how she ought to act.

5 Lessons, Objections, and Replies

Because the value of picking from the menu $\{f, g, \$2\frac{1}{8}\}$ depends on the procedure that's used, it lacks a stable value. But there should be a univocal answer about whether rationality requires Reu to pick from that menu (which is what's behind Door #1) or accept the \$2 1/16 (which is what's behind Door #2). And if no stable value can be assigned to picking from the menu, there will not be a univocal answer. Therefore, we should reject REUT.

That's, very roughly, the argument I've raised against REUT. Is it right? Let's look at some objections and replies.

Response 1: It's not true that picking from the menu $\{f, g, \$2\frac{1}{8}\}$ lacks a stable value. That doesn't follow from the fact that its value depends on the procedure that will be (or, rather, that Reu *believes* will be) used.

1.1 The value of picking from the menu is given, externally, by how the decision-tree looks. Different decision-trees correspond to different problems. So there's nothing objectionable about violating *Agenda Invariance* because there's nothing objectionable about treating *different* decision problems differently.

Response to 1.1: The decision-tree is merely a representation—one that, while subject to external constraints, isn't fully determined by your external circumstances. We aren't imagining that Reu is literally facing external choice-points—like actual forks in an actual road. Rather, the different decision-trees are meant to represent different ways she might structure her internal *deliberation* among the options.

1.2 Just as Reu might be uncertain about which of the items will be picked from the menu were she faced with having to select an item from it (call this a *first-order decision*), Reu should also be uncertain about which of the procedures she will employ (call this a *second-order decision*). Likewise, if one's uncertain about which of the procedures one will employ—if, for example, the value of using a procedure depends on how the agenda is set—one should also be uncertain about *that*. And so on and so forth. The value of the picking situation can be found by iterating this process until it terminates in some stable value.

Response to 1.2: That's true only if, in the limit, a stable evaluation emerges. There's no guarantee of that. In fact, in this case, there won't be: iterating *Process of Elimination* won't result in a stable value assignment. The higher-order decision (about how to set the agenda) merely recapitulates the first-order decision.

Response 2: It's not a mark against REUT that there's no univocal answer about what rationality requires of you in these situations. Sometimes, there's just no fact of the matter about what you are rationally required to do.

Response to 2: This might not be so bad a response if the cases in which there is no fact of the matter about what you are rationally required to do are relatively rare or outlandish. But that's not the case here. Whenever an agent is indifferent between risky gambles, it will be possible to construct—or just wander one's way into—a situation like this one. This is, then, at best, a position of last resort.

Response 3: The value of picking from the menu $\{f, g, \$2^1/_8\}$ does *not* depend on the procedure that's used. You shouldn't treat your future decisions as something to be predicted, but rather as something to be *decided:* pick the *plan* that you like best (where a 'plan' is a complete path through a decision-tree). In this case, you're indifferent between all of them.

> *Response to 3:* This is to endorse, what is sometimes called, Resolute Choice.[11] One should settle on the overall plan that one most desires, and then stick to it—even if that involves acting counter to your preferences in the future. However, if REUT avails itself of this response, then, as Thoma [2019] convincingly argues, its recommendations will be approximately the same as expected utility theory's. This move, then, is at best a Pyrrhic victory. It rescues REUT from objection, but in so doing bleeds it of its distinctive content.

6 Conclusion

I've argued that REUT has some counterintuitive consequences regarding cases of tie-breaking. It doesn't offer clear recommendations about what is rationally required of you when deciding between options you're indifferent between. REUT is meant to be an all-encompassing theory of rational action. This should include decisions in which a source of uncertainty is your own future indecisiveness. REUT, however, cannot handle all such cases in plausible way—it appears to violate either *Menu Superiority* or *Agenda Invariance* or both.

Furthermore, I've raised some worries that REUT doesn't offer an appropriate measure of riskiness to begin with. So, while it might be true that EUT fails to adequately account for differing attitudes toward risk, REUT isn't the appropriate remedy. I have no suggestion for what would be an appropriate remedy. But let's not rue a cure that's worse than the disease.

A Calculations

Let's look at the risk-weighted expected value of deciding between each of the pairs. First, consider deciding between gamble f and its sure-thing cash equivalent $\$f$, which—because Reu has no reason to think she is more likely to select the one rather than the other—corresponds to a 50/50 lottery between the two.

Gamble Between f and \$f

	Pick f		Pick $\$f$	
	E	$\neg E$	E	$\neg E$
$f \oplus_{1/2} \$f$	4	2	$2^1/_8$	$2^1/_8$

[11] See McClennen [1990, 1997] for the classic presentations of the view.

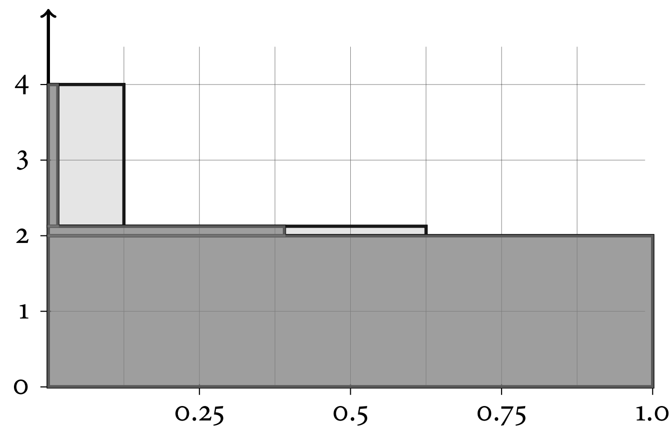


Figure 4: The risk-weighted expected utility of a fifty-fifty lottery between gamble *f* and its sure-thing utility equivalent \$*f* (2.078125).

$$REU\left(f \oplus_{\frac{1}{2}} \$f\right) = 2 + \left(\frac{5}{8}\right)^2 \cdot \left(2\tfrac{1}{8} - 2\right) + \left(\frac{1}{8}\right)^2 \cdot \left(4 - 2^1/8\right)$$
$$= 2 + \frac{25}{512} + \frac{15}{512} = 2^5/64$$

If Reu thinks it's equally likely, when picking between *f* and \$*f*, that she'll end up with either, then picking between the two is valued at \$2⁵⁄64. (See Figure 4.)

Next, consider the decision between gamble *g* and its sure-thing cash equivalent \$*g* (which, again, because Reu is indifferent between *f* and *g*, is identical to \$*f*).

Gamble Between g and \$g

	Pick *g*		Pick \$*g*	
	E	$\neg E$	E	$\neg E$
$g \oplus_{1/2} \$g$	1	3	2¹⁄8	2¹⁄8

$$REU\left(g \oplus_{1/2} \$g\right) = 1 + \left(\frac{7}{8}\right)^2 \cdot \left(2^1/8 - 2\right) + \left(\frac{3}{8}\right)^2 \cdot \left(3 - 2\tfrac{1}{8}\right)$$
$$= 1 + \frac{441}{512} + \frac{63}{512} = 1^{63}/64$$

If Reu thinks it's equally likely, when picking between *g* and \$*g*, that she'll end up with either, then picking between the two is valued at \$1⁶³⁄64. (See Figure 5.)

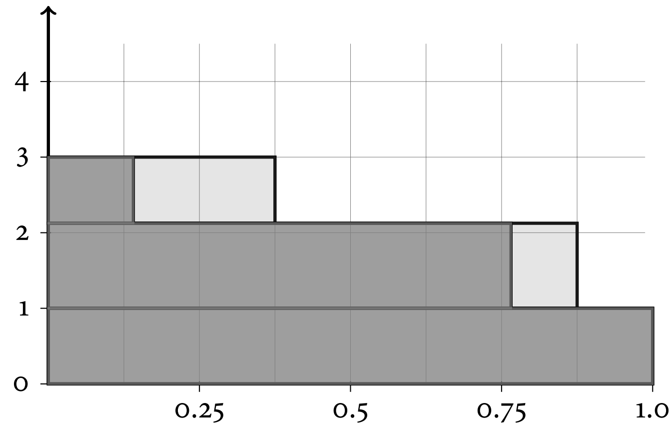


Figure 5: The risk-weighted expected utility of a fifty-fifty lottery between gamble *g* and its sure-thing utility equivalent \$*g* (1.984375).

References

Lara Buchak. *Risk and Rationality*. Oxford University Press, 2013.

Harry Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

Edward F. McClennen. *Rationality and Dynamic Choice: Foundational Explorations*. Cambridge University Press, 1990.

Edward F. McClennen. Pragmatic rationality and rules. *Philosophy and Public Affairs*, 26(3):210–258, 1997.

Michael Rothschild and Joseph Stiglitz. Increasing risk: I. a definition. *Journal of Economic Theory*, 2(3):225–243, 1970.

Johanna Thoma. Risk aversion and the long run. *Ethics*, 129(2), 2019.

Amos Tversky and Daniel Khaneman. Prospect theory: An analysis of decision under risk. *Econometrica*, 47:263–291, 1979.

Amos Tversky and Daniel Khaneman. Choice, values, and frames. *American Psychologist*, 39:341–350, 1984.

Amos Tversky and Daniel Khaneman. Rational choice and the framing of decisions. *The Journal of Business*, 59(4 Part 2: The Behavioral Foundations of Economic Theory):S251–S278, 1986.

Edna Ullmann-Margalit and Sidney Morgenbesser. Picking and choosing. *Social Research*, 44(4):757–785, 1977.